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Maximal Gains of Antennas

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Abstract

The gain of an antenna is defined as the ratio of the radiated energy $\mathcal{E}$ to the heat loss H. For a linear, finite antenna of length 2ℓ, this gain can be shown to have a least upper bound B for all possible current distributions and for all ℓ. We determine B explicitly. For a fixed small value of ℓ, the maximal gain is computed up to an error term of order ℓ^3. If the gain of an antenna is defined as the ratio of the energy $\mathcal{E}^*$ radiated into a half-space, to the total heat loss H, similar but less complete results can be derived. Finally, it is shown that the problem of maximizing $\mathcal{E}^*/\mathcal{E}$ does not have a solution. There exists no realizable current distribution on the antenna which maximizes this quotient.

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1. Introduction and Summary

It is well known and easily proved that we can approximate any arbitrary radiation pattern by using an antenna having a simple shape as, for instance, a spherical antenna, if we can generate any prescribed distribution of current on the antenna. To do this is a difficult technical problem which we shall not investigate here. Another fact which is almost trivial is this: let ϕ,θ be the angles in a system of polar coordinates and let

$$P(\phi,\theta)$$

be a radiation pattern. This means that for large distances r from the antenna, a particular component of the electromagnetic field is of the form

$$\frac{e^{ikr}}{r} P(\phi,\theta)$$

Then $P(\phi,\theta)$ is an analytic function of ϕ,θ provided that the antenna is a finite one and that the current distribution is an absolutely integrable function of the coordinates on the antenna. It follows from this fact that we cannot produce a radiation pattern P such that P is identically zero in a cone with a finite opening as, for instance, in a half space. However, we still can approximate such a non-analytic radiation pattern to any degree of accuracy if we are free to choose the current distribution on the antenna as we wish. Under these circumstances, it seems impossible to define an "optimal" current distribution.

Now it has been observed long ago by K. Fraenz (and probably by others) that any discussion of "optimal" current distribution should

include the heat losses as a gauge in terms of which the " gain" (no matter how defined) should be measured. Following this remark, we shall investigate the ratios

ρ = total radiated energy to heat losses.

ρ^* = energy radiated into a half space to heat losses for a straight linear antenna of finite length 2ℓ.

We shall show that, for a given ℓ and a given wave number k, both ρ and ρ^* have well defined maxima. Their dependency on ℓ will be investigated, particularly for small ℓ and for $\ell \to \infty$. Finally, we shall discuss the problems which arise if we try to find a maximum for $\sigma = \rho^*/\rho$.

The maxima of ρ and ρ^* for a given k will be computed as the first eigenvalues of certain Fredholm integral equations (3.6) and (7.3), respectively. For this reason, the actual maximum of ρ will be denoted by $\lambda_1(\ell)$ and the maximum of ρ^* will be denoted by $\lambda_1^*(\ell)$. For a fixed k and a given ℓ, we find the following results:

If $k\ell$ is small compared to 1, then

$$\lambda_1(\ell) = A\left[\frac{2}{3}k\ell - \frac{2}{45}(k\ell)^3 + O(k\ell)^5\right]$$

$$\lambda_1^*(\ell) = A\left[\frac{1}{3}k\ell + \frac{37}{360}(k\ell)^3 + O(k\ell)^5\right]$$

where

$$A = 2k^2(2\pi)^{3/2}/\omega .$$

Both $\lambda_1(\ell) > \lambda_1^*(\ell)$ are monotonically increasing functions of ℓ. They

satisfy the inequalities

$$\lambda_1(\ell) > \lambda_1^*(\ell) > \frac{1}{2}\,\lambda_1(\ell)$$

and the limit relations

$$\lim_{\ell \to \infty} \lambda_1(\ell) = \lim_{\ell \to \infty} \lambda_1^*(\ell) = A\sqrt{2\pi}$$

For small values of $k\ell$, the eigenfunctions (i.e., the current distributions) belonging to λ_1 and λ_1^* can be expanded in power series in $k\ell$ with coefficients which are power series in kz. However, the exact range of convergence of these power series is not known. For large values of $k\ell$, practically any current distribution will produce "almost" the maximal amount of radiated energy; in particular, this will be true if we have either one of the functions

$$\cos(n\,\pi\,z/\ell), \quad i\sin(n\,\pi\,z)/\ell, \qquad (n=1, 2, 3, \ldots)$$

as current distributions. For details, see Section 5. Obviously, no such statement can be true for the current distributions belonging to λ_1^*, unless we take a $k\ell$ which is so large that λ_1^* is practically equal to λ_1.

Whereas the ratio of radiated energy (either into the whole space or into a half space) versus heat losses can be maximized by current distributions which are continuous and even differentiable functions of the coordinate z on the antenna, no such statement can be true if we consider the ratio ρ^*/ρ of the energy radiated into a half space versus the total radiated energy. This will be shown in Section 10. If we try to maximize ρ^*/ρ, either the current distribution

will tend towards a function which is so singular that the usual arguments leading to an integral equation for this function cannot be applied, or any sequence of current distributions which produces a monotonically increasing sequence of values of ρ^*/ρ will be such that the values of both ρ^* and ρ tend towards zero.

2. Notations and basic formulas.

Consider the time-periodic linear current distribution, I, given by the function

$$I = f(z)e^{-i\omega t}$$

where $f(z) = \alpha(z) + i\beta(z)$; $i = \sqrt{-1}$; ω is the (constant) frequency; x, y, z and t, Cartesian space coordinates and time respectively.

We make the following assumptions:

$f(z)$ continuous for $-\ell \leq z \leq \ell$

$f(z)$ independent of t

$f(z) \not\equiv 0$ for $-\ell \leq z \leq \ell$

$f(z) \equiv 0$ for $|z| > \ell$

ℓ a real positive constant

We are interested in three quantities associated with this current distribution:

1) $\mathcal{E}$, the energy radiated in vacuum by this current.

2) $\mathcal{E}^*$, the energy radiated by this current in vacuum to upper half space, i.e. $z \geq 0$.

3) $\mathcal{H}$, the heat losses.

1) The Determination of $\mathcal{E}$, the Radiated Energy

Let $$M = \int_{-\ell}^{\ell} \frac{e^{ikR}}{R} f(s)ds$$

where $R = \sqrt{x^2 + y^2 + (z-s)^2}$ and k(wave number) $= \frac{\omega}{c}$,

c the speed of light.

Let $\vec{M} = (0, 0, M)$. Then the vector potential $\vec{N}$ of the electromagnetic field generated by this current will be given by

$$\vec{N} = e^{-i\omega t} \vec{M}$$

Then the electric field $\vec{E}$ and magnetic field $\vec{H}$ will be given, respectively, by

$$\vec{E} = \frac{i}{\omega} \text{ curl curl } \vec{N}, \text{ and } \vec{H} = \text{curl } \vec{N} \quad .$$

The following four steps are an outline for the procedure for finding the energy radiated per second:

(i) Consider the time-dependent Poynting vector

$$\vec{S}(t) = 2\text{Re}\vec{E} \times \text{Re}\vec{H} \quad . \qquad (\times : \text{vector product})$$

Let $\vec{S}$ be the time-average of $\vec{S}(t)$ so that

$$\vec{S} = \lim_{T \to \infty} \frac{1}{T} \int_0^T 2(\text{Re}\vec{E} \times \text{Re}\vec{H})\, dt$$

(ii) Let S_n be the radial component of $\vec{S}$.

Then

$$S_n = \frac{x}{r} S_1 + \frac{y}{r} S_2 + \frac{z}{r} S_3$$

where $r = \sqrt{x^2 + y^2 + z^2}$ and $\vec{S} = (S_1, S_2, S_3)$.

(iii) Let A_ρ be a large sphere with radius ρ, and center at the origin.

Form the surface integral

$$\iint_{A_\rho} S_n \, dA_\rho = \mathcal{E}_\rho$$

(iv) Take $\lim_{\rho \to \infty} \mathcal{E}_\rho = \mathcal{E}$, the radiated energy.

We carry out step (i) as follows:

Writing

$$\vec{E} = e^{-i\omega t}\left(\frac{i}{\omega} \text{ curl curl } \vec{M}\right), \quad \vec{H} = e^{-i\omega t}(\text{curl } \vec{M})$$

and letting

$$\frac{i}{\omega} \text{ curl curl } \vec{M} = \vec{E}_1 + i\vec{E}_2 , \quad \text{curl } \vec{M} = \vec{H}_1 + i\vec{H}_2$$

we have

$$\vec{E} = e^{-i\omega t}(\vec{E}_1 + i\vec{E}_2) \text{ and } \vec{H} = e^{-i\omega t}(\vec{H}_1 + i\vec{H}_2) ,$$

so that

$$2\text{Re}\vec{E} \times \text{Re}\vec{H} = 2\vec{E}_1 \times \vec{H}_1 \cos^2\omega t + 2\vec{E}_2 \times \vec{H}_2 \sin^2\omega t$$

$$+ 2(\vec{E}_1 \times \vec{H}_2 + \vec{E}_2 \times \vec{H}_1) \sin \omega t \cos \omega t .$$

Then $\frac{1}{T} \int_0^T 2(\text{Re}\vec{E} \times \text{Re}\vec{H}) \, dt$

$$= \left(1 + \frac{\sin \omega T \cos \omega T}{\omega T}\right) \vec{E}_1 \times \vec{H}_1$$

$$+ \left(1 - \frac{\sin \omega T \cos \omega T}{\omega T}\right) \vec{E}_2 \times \vec{H}_2$$

$$+ \frac{\sin^2 \omega T}{\omega T} (\vec{E}_1 \times \vec{H}_2 + \vec{E}_2 \times \vec{H}_1) .$$

Letting $T \to \infty$ in this sum we get

$$\vec{S} = \vec{E}_1 \times \vec{H}_1 + \vec{E}_2 \times \vec{H}_2 .$$

But $\vec{E}_1 \times \vec{H}_1 + \vec{E}_2 \times \vec{H}_2 = \mathrm{Re}\left\{ (\vec{E}_1 + i\vec{E}_2) \times (\vec{H}_1 - i\vec{H}_2) \right\}$

so that

$$\vec{S} = \mathrm{Re}\left\{ (\frac{i}{\omega} \text{ curl curl } \vec{M}) \times \overline{(\text{curl } \vec{M})} \right\}$$ (overhead bar indicates complex conjugate)

Step (ii). In this section, subscripts x, y, z means partial differentiation with respect to these variables.

$$\text{curl } \vec{M} = (M_y, M_x, 0) ,$$

$$\frac{i}{\omega} \text{ curl curl } \vec{M} = \frac{i}{\omega} \left[M_{xz} , M_{yz}, -(M_{xx} + M_{yy}) \right]$$

and

$$\vec{S} = \mathrm{Re}\left\{ (\frac{i}{\omega} \text{ curl curl } \vec{M}) \times \overline{(\text{curl } \vec{M})} \right\}$$

$$= \mathrm{Re}\left\{ - \frac{i}{\omega} \left[\overline{M}_x (M_{xx} + M_{yy}), \overline{M}_y (M_{xx} + M_{yy}), \overline{M}_x M_{xz} + \overline{M}_y M_{yz} \right] \right\}$$

so that

$$S_n = \mathrm{Re}\left\{ - \frac{i}{\omega} \left[\frac{x}{r} \overline{M}_x (M_{xx} + M_{yy}) + \frac{y}{r} \overline{M}_y (M_{xx} + M_{yy}) + \frac{z}{r} (\overline{M}_x M_{xz} + \overline{M}_y M_{yz}) \right] \right\} .$$

We now compute the derivatives occuring in S_n,

$$R_x = \frac{x}{R} ; \quad R_y = \frac{y}{R} ; \quad R_z = \frac{z - s}{R} ;$$

$$(e^{ikR})_x = \frac{ikx}{R} e^{ikR}; \quad (e^{ikR})_y = \frac{iky}{R} e^{ikR} ; \quad (e^{ikR})_z = \frac{ik(z-s)}{R} e^{ikR} ;$$

$$(\frac{e^{ikR}}{R})_x = e^{ikR} (\frac{ikx}{R^2} - \frac{x}{R^3})$$

$$(\frac{e^{ikR}}{R})_y = e^{ikR} (\frac{iky}{R^2} - \frac{y}{R^3})$$

$$(\frac{e^{ikR}}{R})_z = e^{ikR} (\frac{ik(z-s)}{R^2} - \frac{(z-s)}{R^3})$$

$$\left(\frac{e^{ikR}}{R}\right)_{xx} = e^{ikR}\left(-\frac{3iK}{R^4}x^2 - \frac{k^2}{R^3}x^2 + \frac{3}{R^5}x^2 + \frac{ik}{R^2} - \frac{1}{R^3}\right)$$

$$\left(\frac{e^{ikR}}{R}\right)_{yy} = e^{ikR}\left(-\frac{3ik}{R^4}y^2 - \frac{k^2}{R^3}y^2 + \frac{3}{R^5}y^2 + \frac{ik}{R^2} - \frac{1}{R^3}\right)$$

$$\left(\frac{e^{ikR}}{R}\right)_{xz} = e^{ikR}\left(-\frac{3ik}{R^4}x(z-s) - \frac{k^2}{R^3}x(z-s) + \frac{3}{R^5}x(z-s)\right)$$

$$\left(\frac{e^{ikR}}{R}\right)_{yz} = e^{ikR}\left(-\frac{3ik}{R^4}y(z-s) - \frac{k^2}{R^3}y(z-s) + \frac{3}{R^5}y(z-s)\right).$$

Let $a = e^{ikR}\left(-\frac{3ik}{R^4} - \frac{k^2}{R^3} + \frac{3}{R^5}\right)$ and

$$b = e^{ikR}\left(+\frac{ik}{R^2} - \frac{1}{R^3}\right)$$

then

$$\left(\frac{e^{ikR}}{R}\right)_x = bx \; ; \; \left(\frac{e^{ikR}}{R}\right)_y = by$$

$$\left(\frac{e^{ikR}}{R}\right)_{xx} = ax^2 + b; \quad \left(\frac{e^{ikR}}{R}\right)_{yy} = ay^2 + b$$

$$\left(\frac{e^{ikR}}{R}\right)_{xz} = ax(z-s); \; \left(\frac{e^{ikR}}{R}\right)_{yz} = ay(z-s) \quad .$$

Let

$$P(a) = \int_{-\ell}^{\ell} f(s)a \, ds \quad , \quad P(b) = \int_{-\ell}^{\ell} f(s)b \, ds, \text{ and } P_1(a) = \int_{-\ell}^{\ell} sf(s)a \, ds \quad .$$

Then

$$M_{xx} = \int_{-\ell}^{\ell} f(s)\left(\frac{e^{ikR}}{R}\right)_{xx} ds = \int_{-\ell}^{\ell} (ax^2 + b)f(s) \, ds = x^2P(a) + P(b)$$

$$M_{yy} = \int_{-\ell}^{\ell} f(s)\left(\frac{e^{ikR}}{R}\right)_{yy} ds = \int_{-\ell}^{\ell} (ay^2 + b)f(s) \, ds = y^2P(a) + P(b)$$

$$M_{xz} = \int_{-\ell}^{\ell} f(s)\left(\frac{e^{ikR}}{R}\right)_{xz} ds = \int_{-\ell}^{\ell} ax(z-s)f(s) \, ds = xzP(a) - xP_1(a)$$

$$M_{yz} = \int_{-\ell}^{\ell} f(s)\,\left(\frac{e^{ikR}}{R}\right)_{yz} ds = \int_{-\ell}^{\ell} ay(z-s)f(s)\,ds = yzP(a) - yP_1(a)$$

$$M_x = \int_{-\ell}^{\ell} f(s)\,\left(\frac{e^{ikR}}{R}\right)_{x} ds = \int_{-\ell}^{\ell} bx\, f(s)\,ds = xP(b)$$

$$M_y = \int_{-\ell}^{\ell} f(s)\,\left(\frac{e^{ikR}}{R}\right)_{y} ds = \int_{-\ell}^{\ell} by\, f(s)\,ds = yP(b).$$

We then have

$$\frac{x}{r}\,\overline{M}_x(M_{xx}+M_{yy}) = \frac{x^2(x^2+y^2)}{r}\,P(a)\,\overline{P(b)} + \frac{2x^2}{r}\,P(b)\,\overline{P(b)}$$

$$\frac{y}{r}\,\overline{M}_y(M_{xx}+M_{yy}) = \frac{y^2(x^2+y^2)}{r}\,P(a)\,\overline{P(b)} + \frac{2y^2}{r}\,P(b)\,\overline{P(b)}$$

$$\frac{z}{r}\,\overline{M}_x\,M_{xz} = \frac{z^2x^2}{r}\,P(a)\,\overline{P(b)} - \frac{zx^2}{r}\,P_1(a)\,\overline{P(b)}$$

$$\frac{z}{r}\,\overline{M}_y\,M_{yz} = \frac{z^2y^2}{r}\,P(a)\,\overline{P(b)} - \frac{zy^2}{r}\,P_1(a)\,\overline{P(b)}$$

but the sum of the terms on the left hand side of these equations constitutes the bracket in S_n . Adding the right hand sides of these equations we get

$$\frac{(x^2+y^2+z^2)(x^2+y^2)}{r}\,P(a)\,\overline{P(b)} + 2\frac{(x^2+y^2)}{r}\,|P(b)|^2 - \frac{z(x^2+y^2)}{r}\,P_1(a)\,\overline{P(b)} \quad .$$

Letting

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\phi$$

this sum becomes

$$r^3 \sin^2\theta\, P(a)\,\overline{P(b)} + 2r \sin^2\theta\, |P(b)|^2 - r^2\sin^2\theta \cos\phi\, P_1(a)\,\overline{P(b)}$$

so that

$$S_n = \mathrm{Re}\left\{ -\frac{i}{\omega}\left[r^3\sin^2\theta\, P(a)\,\overline{P(b)} + 2r \sin^2\theta\, |P(b)|^2 - r^2\sin^2\theta \cos\phi P_1(a)\,\overline{P(b)} \right] \right\}.$$

Step (iii) Substituting spherical coordinates in the definition for $\mathcal{E}_\rho$ and letting $\rho = r$ gives

$$\mathcal{E}_r = \int_0^{\pi}\int_0^{2\pi} r^2 \sin\theta\, S_n \, d\phi\, d\theta$$

$$= \int_0^{\pi}\int_0^{2\pi} \mathrm{Re}\left(\frac{i}{\omega} r^4 \sin^3\theta \cos\phi\, P_1(a)\, \overline{P(b)}\right) d\phi\, d\theta$$

$$+ \int_0^{\pi}\int_0^{2\pi} \mathrm{Re}\left(-\frac{i}{\omega} 2r^3 \sin^3\theta\, |P(b)|^2\right) d\phi\, d\theta$$

$$+ \int_0^{\pi}\int_0^{2\pi} \mathrm{Re}\left(-\frac{i}{\omega} r^5 \sin^3\theta\, P(a)\, \overline{P(b)}\right) d\phi\, d\theta .$$

Step (iv)

Let
$$U_3(s,R) = -k^2 e^{ikR} f(s) ,$$
$$U_4(s,R) = -3ike^{ikR} f(s) ,$$
$$U_5(s,R) = 3e^{ikR} f(s) ,$$
$$V_3(s,R) = -k^2 s\, e^{ikR} f(s) ,$$
$$V_4(s,R) = -3iks\, e^{ikR} f(s) ,$$
$$V_5(s,R) = 3se^{ikR} f(s) ,$$
$$W_2(s,R) = ik\, e^{ikR} f(s) ,$$
and
$$W_3(s,R) = -e^{ikR} f(s) .$$

Then $$P(a) = \sum_{j=3}^{5} \int_{-\ell}^{\ell} \frac{U_j}{R^j}\, ds$$

$$P_1(a) = \sum_{n=3}^{5} \int_{-\ell}^{\ell} \frac{V_n}{R^n}\, ds$$

and $$P(b) = \sum_{m=2}^{3} \int_{-\ell}^{\ell} \frac{W_m}{R^m}\, ds$$

And
$$P(a)\,\overline{P(b)} = \sum_{j,m} \int_{-\ell}^{\ell} \frac{U_j}{R^j}\,ds \int_{-\ell}^{\ell} \frac{W_m}{R^m}\,ds \qquad \begin{pmatrix} j = 3,4,5 \\ m = 2,3 \end{pmatrix}$$

$$\min\,(j + m) = 5, \qquad \text{where } j = 3 \text{ and } m = 2\ ,$$

$$P_1(a)\,\overline{P(b)} = \sum_{n,m} \int_{-\ell}^{\ell} \frac{V_n}{R^n}\,ds \int_{-\ell}^{\ell} \frac{W_m}{R^m}\,ds\ , \qquad \begin{pmatrix} n = 3,4,5 \\ m = 2,3 \end{pmatrix}$$

$$\min(k + m) = 5$$

$$P(b)\,\overline{P(b)} = \sum_{j',j} \int_{-\ell}^{\ell} \frac{W_j}{R^j}\,ds \int_{-\ell}^{\ell} \frac{\overline{W_{j'}}}{R^{j'}}\,ds\ , \qquad \begin{pmatrix} j = 2,3 \\ j = 2,3 \end{pmatrix}$$

$$\min\,(j + j') = 4\ .$$

Let $\mathcal{E}_r = Q_1 + Q_2 + Q_3$

where
$$Q_1 = \int_0^{\pi} \int_0^{2\pi} \mathrm{Re}\left(\frac{i}{\omega}\, r^4 \sin^3\theta \cos\phi\ P_1(a)\,\overline{P(b)} \right) d\phi\ d\theta$$

$$Q_2 = \int_0^{\pi} \int_0^{2\pi} \mathrm{Re}\left(-\frac{i}{\omega}\, 2r^3 \sin^3\theta\ |P(b)|^2 \right) d\phi\ d\theta$$

$$Q_3 = \int_0^{\pi} \int_0^{2\pi} \mathrm{Re}\left(-\frac{i}{\omega}\, r^5 \sin^3\theta\ P(a)\,\overline{P(b)} \right) d\phi\ d\theta$$

then $\lim_{r\to\infty} \mathcal{E}_r = \lim_{r\to\infty} Q_1 + \lim_{r\to\infty} Q_2 + \lim_{r\to\infty} Q_3$.

We note the following statements:

As $r \to \infty$

A) $R = r - s\cos\theta + o(\frac{1}{r})$

B) $$\begin{cases} e^{ikR} = e^{ikr}\, e^{-iks\cos\theta}\, e^{o(\frac{1}{r})} \\ e^{-ikR} = e^{-ikr}\, e^{iks\cos\theta}\, e^{o(\frac{1}{r})} \end{cases}$$

C) $\lim \frac{r^n}{R^n} = 1$, $\lim \frac{1}{R^j} = 0 \qquad j > 0$.

D) If $j < k + m$, $g(s,R)$ and $h(s,R)$ uniformly bounded for $-\ell \leq s \leq \ell$ and large R, then

$$\lim r^j \int_{-\ell}^{\ell} \frac{g(s,R)}{R^k}\, ds \int \frac{h(s,R)}{R^m}\, ds = 0 .$$

The expressions $r^4 P_1(a)\ \overline{P(b)}$ and $r^3 P(b)\ \overline{P(b)}$ consist of sums of terms satisfying the conditions of statement D). Hence

$$\lim_{r\to\infty} Q_1 = \lim_{r\to\infty} Q_2 = 0$$

The expression $r^5\ P(a)\ \overline{P(b)}$ consists of a sum of terms, all of which satisfy the conditions of statement D), with one exception, and hence have the limit zero as $r \to \infty$. The single exception is

$$r^5 \int_{-\ell}^{\ell} \frac{U_3(s,R)}{R^3}\, ds \int_{-\ell}^{\ell} \frac{\overline{W_2(s,R)}}{R^2}\, ds$$

$$\text{or } r^5 \int_{-\ell}^{\ell} \frac{-k^2 e^{ikR} f(s)}{R^3}\, ds \int_{-\ell}^{\ell} \frac{(-ik)\, e^{-ikR}\, \overline{f(s)}}{R^2}\, ds$$

$$= ik^3 \int_{-\ell}^{\ell} \frac{r^3}{R^3} e^{ikR} f(s)\, ds \int_{-\ell}^{\ell} \frac{r^2}{R^2} e^{-ikR}\, \overline{f(s)}\, ds$$

$$= ik^3 \int_{-\ell}^{\ell} \frac{r^3}{R^3} e^{o(\frac{1}{r})} e^{-iks\cos\theta} f(s)\,ds \int_{-\ell}^{\ell} \frac{r^2}{R^2} e^{o(\frac{1}{r})} e^{iks\cos\theta}\,\overline{f(s)}\,ds$$

by statement B). By statement C) we get

$$\lim_{r\to\infty} \frac{r^3}{R^3} e^{o(\frac{1}{r})} = \lim_{r\to\infty} \frac{r^2}{R^2} e^{o(\frac{1}{r})} = 1 \quad .$$

Therefore

$$\lim_{r\to\infty} r^5 \int_{-\ell}^{\ell} \frac{U_3(s_1R)}{R^3}\,ds \int_{-\ell}^{\ell} \frac{\overline{W_2(s_1R)}}{R^2}\,ds$$

$$= ik^3 \int_{-\ell}^{\ell} e^{-iks\cos\theta} f(s)\,ds \int_{-\ell}^{\ell} e^{iks\cos\theta}\,\overline{f(s)}\,ds \quad .$$

So that

$$\mathcal{E} = \lim_{r\to\infty} \mathcal{E}_r = \lim_{r\to\infty} Q_3$$

$$= \int_0^{\pi}\int_0^{2\pi} \mathrm{Re}\left[\frac{k^3}{\omega}\sin^3\theta \int_{-\ell}^{\ell} e^{-iks\cos\theta} f(s)\,ds \overline{\int_{-\ell}^{\ell} e^{-iks\cos\theta} f(s)\,ds}\right] d\phi\,d\theta$$

$$= 2\pi\frac{k^3}{\omega}\int_0^{\pi} \sin^3\theta \left\{\int_{-\ell}^{\ell} e^{-iks\cos\theta} f(s)\,ds \overline{\int_{-\ell}^{\ell} e^{-iks\cos\theta} f(s)\,ds}\right\} d\theta$$

Letting s = u in the first bracketed integral and s = v in the second bracketed integral, we get

$$\int_{-\ell}^{\ell} e^{-iku\cos\theta} f(u)\,du \int_{-\ell}^{\ell} \overline{e^{-ik\cos a} f(v)}\,dv$$

$$= \int_{-\ell}^{\ell}\int_{-\ell}^{\ell} e^{-ik\cos\theta(u-v)} f(u)\,\overline{f(v)}\,du\,dv$$

$$= \int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \Big[\cos\Big[(u-v)k\cos\theta\Big]\Big(\alpha(u)\alpha(v)+\beta(u)\beta(v)\Big) - \sin\Big[(u-v)k\cos\theta\Big]\Big(\alpha(u)\beta(v)-\alpha(v)\beta(u)\Big)\Big]\,du\,dv \quad .$$

Substituting back and changing order of integration we get

$$\mathcal{E} = 2\frac{\pi k^3}{\omega}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\left[\alpha(u)\alpha(v)+\beta(u)\beta(v)\right]\left[\int_0^{\pi}\sin^3\theta\,\cos\left[(u-v)k\,\cos\theta\right]d\theta\right]du\,dv$$
$$-\frac{2\pi k^3}{\omega}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\left[\alpha(u)\beta(v)-\alpha(v)\beta(u)\right]\left[\int_0^{\pi}\sin^3\theta\,\sin\left[(u-v)k\,\cos\theta\right]d\theta\right]du\,dv\ .$$

We note that

$$\int_0^{\pi/2}\sin^3(\gamma)\,\cos(\zeta\,\cos\gamma)\,d\gamma = \sqrt{2\pi}\,\frac{J_{3/2}(\zeta)}{\zeta^{3/2}}$$

and $$\int_0^{\pi/2}\sin^3(\gamma)\,\sin(\zeta\,\cos\gamma)\,d\gamma = \sqrt{2\pi}\,\frac{\mathbf{H}_{3/2}(\zeta)}{\zeta^{3/2}}$$

where J and $\mathbf{H}$ represent the Bessel and Struve functions respectively.

It is clear that

$$\int_{\pi/2}^{\pi}\sin^3(\gamma)\,\sin(\zeta\,\cos\gamma)\,d\gamma = -\int_0^{\pi/2}\sin^3(\gamma)\,\sin(\zeta\,\cos\gamma)\,d\gamma$$

and

$$\int_{\pi/2}^{\pi}\sin^3(\gamma)\,\cos(\zeta\,\cos\gamma)\,d\gamma = \int_0^{\pi/2}\sin^3(\gamma)\,\cos(\zeta\,\cos\gamma)\,d\gamma$$

Therefore

$$\int_0^{\pi}\sin^3\theta\,\cos\left[(u-v)k\,\cos\theta\right]\,d\theta = 2\sqrt{2\pi}\,\frac{J_{3/2}\left(k(u-v)\right)}{\left(k(u-v)\right)^{3/2}}$$

and

$$\int_0^{\pi}\sin^3\theta\,\sin\left[(u-v)k\,\cos\theta\right]\,d\theta = 0\ .$$

So that we get for the radiated energy,

$$\mathcal{E} = \frac{2(2\pi)^{3/2}k^3}{\omega} \int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \frac{J_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}}\Big(\alpha(u)\alpha(v) + \beta(u)\beta(v)\Big)\, du\, dv \ .$$

2) Computation of $\mathcal{E}^*$, the energy radiated to upper half space.

The four-step procedure outlined for computing $\mathcal{E}$ is repeated identically with one exception. In step (iii) instead of forming the surface integral with A_ρ, a sphere of radius ρ and center at the origin, we now form the surface integral with A_ρ^*, the half of this sphere lying above the x,y plane. In carrying out this computation for $\mathcal{E}^*$ this means only the difference of taking the limits of integration with respect to Θ to be $\frac{\pi}{2}$ to π, instead of 0 to $\frac{\pi}{2}$.

We therefore get

$$\mathcal{E}^* = \frac{2\pi k^3}{\omega}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\Big[\alpha(u)\alpha(v) + \beta(u)\beta(v)\Big]\left[\int_{\pi/2}^{\pi}\sin^3\Theta\,\cos\Big[(u-v)k\cos\Theta\Big]d\Theta\right]du\,dv$$

$$-\frac{2\pi k^3}{\omega}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\Big[\alpha(u)\beta(v) - \alpha(v)\beta(u)\Big]\left[\int_{\pi/2}^{\pi}\sin^3\Theta\,\sin\Big[(u-v)k\cos\Theta\Big]d\Theta\right]du\,dv$$

Substituting from above we have therefore the energy radiated to upper half space,

$$\mathcal{E}^* = \frac{(2\pi)^{3/2}k^3}{\omega}\int_{-\ell}^{\ell}\int_{-\ell}^{\ell}\left\{\frac{J_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}}\Big(\alpha(u)\alpha(v) + \beta(u)\beta(v)\Big)\right.$$

$$\left. + \frac{\mathbf{H}_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}}\Big(\alpha(u)\beta(v) - \alpha(v)\beta(u)\Big)\right\}du\,dv \ .$$

3) Definition of $\mathcal{H}$, the heat loss

We take the heat loss to be

$$C \int_{-\ell}^{\ell} f(s)\ \overline{f(s)}\ ds$$

where C is a constant depending on the carrier (antenna) of the current. We arbitrarily set C = 1 so that we can write

$$\mathcal{H} = \int_{-\ell}^{\ell} \left[\alpha^2(s) + \beta^2(s)\right] ds$$

Having defined the functionals $\mathcal{E}(f)$, $\mathcal{E}^*(f)$ and $\mathcal{H}(f)$, we now pose the following three problems:

I, Find $\rho(\ell)$, where $\rho(\ell) = \max_f \dfrac{\mathcal{E}(f)}{\mathcal{H}(f)}$

II, Find $\rho^*(\ell)$, where $\rho^*(\ell) = \max_f \dfrac{\mathcal{E}^*(f)}{\mathcal{H}(f)}$

and III, Find $\sigma(\ell)$, where $\sigma(\ell) = \max_f \dfrac{\mathcal{E}^*(f)}{\mathcal{E}(f)}$.

3. Integral equations for the current-distribution.

The radiated energy $\mathcal{E}$ and the heat loss $\mathcal{H}$ are functionals of the current-distribution $f(s) = \alpha(s) + i\beta(s)$. If we write

$$(3.1)\qquad \mathcal{E}(f) = \int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \gamma\ \frac{J_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}} \left[\alpha(u)\alpha(v) + \beta(u)\beta(v)\right] du\ dv \quad ,$$

$$(3.2)\qquad \mathcal{H}(f) = \int_{-\ell}^{\ell} \left[\alpha^2(u) + \beta^2(u)\right] du \quad ,$$

where

$$(3.3)\qquad \gamma = \frac{2}{\omega} (2\pi)^{3/2} k^3 \quad .$$

Then the problem of maximizing $\frac{\mathcal{E}(f)}{\mathcal{H}(f)}$ over the class of functions, $\alpha(s) + i\beta(s) = f(s)\ \varepsilon L^2(-\ell,\ell)$ (s real), is the same as maximizing $\mathcal{E}(f)$ subject to the condition $\mathcal{H}(f)$ constant, over the same class of functions.

Setting the first variations equal to zero yields the equations

$$(3.4)\qquad \int_{-\ell}^{\ell} \gamma \frac{J_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}}\,\alpha(u)\,du = \lambda\alpha(v)$$

$$(3.5)\qquad \int_{-\ell}^{\ell} \gamma \frac{J_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}}\,\beta(u)\,du = \lambda\beta(v)$$

We note that both α and β satisfy the equation

$$(3.6)\qquad \lambda\phi(v) = \int_{-\ell}^{\ell} \gamma \frac{J_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}}\,\phi(u)\,du$$

If in (3.6) we let $u = \ell x$ and $v = \ell y$ we get

$$(3.7)\qquad \lambda f(y) = \int_{-1}^{1} \gamma\ell \frac{J_{3/2}\left[k\ell(x-y)\right]}{\left[k\ell(x-y)\right]^{3/2}}\,f(x)dx$$

where $f(x) = \phi(\ell x)$ and $f(y) = \phi(\ell y)$.

We note that

$$\int_{-\infty}^{\infty} e^{ivs}\,\gamma \frac{J_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}}\,dv = \left\{\begin{array}{ll} \frac{\gamma}{k^3}\sqrt{\frac{\pi}{2}}\,e^{ius}(k^2-s^2), & |s| \le k \\ 0 & |s| > k \end{array}\right\}$$

If we let $\int_{-\infty}^{\infty} e^{ivs}\,\phi(v)\,dv = g(s)$

then

$$\phi(u) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itu}\, g(t)\, dt$$

Therefore operating on both sides of (3.6) with

$$\int_{-\infty}^{\infty} e^{ivs} \left\{ \qquad \right\} dv$$

we get

$$\lambda g(s) = \frac{\gamma}{k^3} \frac{1}{2\sqrt{2\pi}} \int_{-\ell}^{\ell} \left[\left\{ \begin{array}{ll} e^{ius}(k^2 - s^2), & |s| \leq k \\ 0 & |s| > k \end{array} \right\} \int_{-\infty}^{\infty} e^{-itu}\, g(t)\, dt \right] du$$

Noting that $\int_{-\ell}^{\ell} e^{iu(s-t)}\, du = 2\, \frac{\sin [\ell(t-s)]}{t-s}$

This equation can be written

$$(3.8) \quad \lambda g(s) = \frac{\gamma}{k^3} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left\{ \begin{array}{ll} \frac{\sin [\ell(t-s)]}{t-s}, & |s| \leq k \\ 0 & |s| > k \end{array} \right\} g(t)\, dt$$

If $\lambda \neq 0$ the equation becomes

$$(3.9) \quad \lambda g(s) = \frac{\gamma}{k^3} \frac{1}{\sqrt{2\pi}} \int_{-k}^{k} \frac{\sin \left[\ell(t-s)\right]}{t-s} (k^2 - s^2)\, g(t)\, dt$$

Letting $t = kx$ and $s = ky$ gives

$$(3.10) \quad \lambda g(ky) = \frac{\gamma}{\sqrt{2\pi}} \int_{-1}^{1} \frac{\sin \left[k\ell(x-y)\right]}{k(x-y)} (1 - y^2)\, g(kx)\, dx$$

Letting $\frac{g(kx)}{\sqrt{1-x^2}} = h(x)$ and $\frac{g(ky)}{\sqrt{1-y^2}} = h(y)$

and noting that $\frac{\sin \zeta}{\zeta} = \sqrt{\frac{\pi}{2}} \frac{J_{1/2}(\zeta)}{\zeta^{1/2}}$

this equation becomes

$$(3.11) \qquad \lambda h(y) = \int_{-1}^{1} \frac{\gamma}{2} \ell \frac{J_{1/2}\left[k\ell(x-y)\right]}{\left[k\ell(x-y)\right]^{1/2}} \sqrt{1-y^2}\ \sqrt{1-x^2}\ h(x)\, dx \,.$$

We note that to every eigenvalue of (3.6), (3.7) and (3.11) there corresponds at least one even or one odd eigenfunction.

We will denote the kernels of (3.6), (3.7) and (3.11) by K, K_1, K_2 respectively. We note that the set of eigenvalues of all three equations is the same. The kernels are all symmetric and analytic in their arguments for $0 > \ell < \infty$. Therefore for any $0 < \ell < \infty$ they have a real discrete spectrum.

We now show that K is a positive definitive kernel for any $0 < \ell < \infty$. We note that

$$\frac{J_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}} = \sum_{n=0}^{\infty} C_n \frac{J_{3/2+n}(ku)}{(ku)^{3/2}} \quad \frac{J_{3/2+n}(kv)}{(kv)^{3/2}}$$

where $C_n = \sqrt{\frac{\pi}{2}}\,(n+1)(n+2)(n+3/2)$.

Therefore for any admissible q(s)

$$\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \gamma \frac{J_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}} q(u)\, q(v)\, du\, dv$$

$$= \gamma \sum_{n=0}^{\infty} C_n \int_{-\ell}^{\ell} \frac{J_{3/2+n}(ku)}{(ku)^{3/2}}\, q(u)\, du \int_{-\ell}^{\ell} \frac{J_{3/2+n}(kv)}{(kv)^{3/2}}\, q(v)\, dv$$

$$= \gamma \sum_{n=0}^{\infty} C_n \left[\int_{-\ell}^{\ell} \frac{J_{3/2+n}(ks)}{(ks)^{3/2}}\, q(s)\, ds\right]^2 \geq 0 .$$

We note again that $\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \gamma \frac{J_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}} q(u)q(v)\, du\, dv$

$$= \delta \int_0^{\pi} \sin^3\theta \left\{\int_{-\ell}^{\ell} e^{isk\cos\theta} q(s)\, ds \int_{-\ell}^{\ell} \overline{e^{isk\cos\theta} q(s) ds}\right\} d\theta ,$$

δ = a positive constant

If we let $k\cos\theta = t$, the right hand side of the equation becomes

$$\frac{\delta}{k} \int_{-k}^{k} \left(1 - \frac{t^2}{k^2}\right) g(t)\, \overline{g(t)}\, dt \quad ,$$

where for $-k \leq t < k$, $g(t) \equiv G(t)$, the Fourier transform of q(s). Since $q(s) \equiv 0$ for $|s| > \ell$, G(t) is analytic. If we set the left hand side of the equation equal to zero it follows that $g(t) \equiv 0$ for $-k \leq t \leq k$, so $G(t) \equiv 0$ for $-k \leq t \leq k$ and therefore identically zero, hence $q(s) \equiv 0$.

It follows that the spectrum of K for $0 < \ell < \infty$ is positive. Therefore K_1 and K_2 are also positive definite kernels.

We now show that the largest eigenvalue is a non-decreasing function of ℓ. For any fixed ℓ, $0 < \ell < a$ and any admissible normalized function $q(s)$

$$\lambda_1(\ell) \geq \int_{-\ell}^{\ell}\int_{-\ell}^{\ell} K(u,v)\, q(u)\, q(v)\, du\, dv$$

where $\lambda_1(\ell)$ is the largest eigenvalue. If $q_1(s)$ is an eigenfunction of λ_1 the equality sign holds. Let $\ell > 0$ be fixed and $\phi_1(\ell',s)$ be an eigenfunction of $\lambda_1(\ell')$. Then

$$\lambda_1(\ell') = \int_{-\ell'}^{\ell'}\int_{-\ell'}^{\ell'} K(u,v)\, \phi(\ell',u)\, \phi_1(\ell',v)\, du\, dv \quad .$$

Let $\ell'' > \ell'$ be fixed and

$$p(\ell'',s) = \begin{cases} \phi_1(\ell',s) & , \quad |s| \leq \ell' \\ 0 & \quad |s| > \ell' \end{cases}$$

then

$$\lambda_1(\ell'') \geq \int_{-\ell''}^{\ell''}\int_{-\ell''}^{\ell''} K(u,v)\, p(\ell'',u)\, p(\ell'',v)\, du\, dv \quad .$$

$$= \int_{-\ell'}^{\ell'}\int_{-\ell'}^{\ell'} K(u,v)\, \phi(\ell',u)\, \phi(\ell',v)\, du\, dv = \lambda_1(\ell') \quad .$$

4. Behavior of the largest eigenvalue $\lambda_1(\ell)$ for small ℓ.

Rewrite equation (3.7) in the form

$$(4.1) \qquad \tilde{\lambda}\, f(y) = \int_{-1}^{1} \gamma\, \frac{J_{3/2}\left[k\ell(x-y)\right]}{\left[k\ell(x-y)\right]^{3/2}}\, f(x)\, dx$$

where

$$\tilde{\lambda} = \frac{\lambda}{\ell}$$

Let

$$\tilde{K}_1 = \gamma \frac{J_{3/2}\left[k\ell(x-y)\right]}{\left[k\ell(x-y)\right]^{3/2}} = \gamma \sum_{n=0}^{\infty} \Theta_{2n}\left[k\ (x-y)\right]^{2n}$$

where $\Theta_{2n} = (-1)^n \left[\frac{1}{(2n+2)!} - \frac{1}{(2n+3)!} \right]$.

$\tilde{K}_1$ is an entire function of k and ℓ for bounded x and y.

Let

$$D_m(\ell) = \int_{-1}^{1}\int_{-1}^{1}\cdots\int_{-1}^{1} \begin{vmatrix} \tilde{K}_1(x_1,x_1) & \tilde{K}_1(x_1,x_2) & \cdots & \tilde{K}_1(x_1,x_m) \\ \tilde{K}_1(x_2,x_1) & \tilde{K}_1(x_2,x_2) & \cdots & \tilde{K}_1(x_2,x_m) \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{K}_1(x_m,x_1) & \tilde{K}_1(x_m,x_2) & \cdots & \tilde{K}_1(x_m,x_m) \end{vmatrix} dx_1 dx_2 \cdots dx_m$$

and

$$D_m(x,y) = \int_{-1}^{1}\int_{-1}^{1}\cdots\int_{-1}^{1} \begin{vmatrix} \tilde{K}_1(x,y) & \tilde{K}_1(x,x_1) & \cdots & \tilde{K}_1(x,x_m) \\ \tilde{K}_1(x_1,y) & \tilde{K}_1(x_1,x_1) & \cdots & \tilde{K}_1(x_1,x_m) \\ \vdots & \vdots & \vdots & \vdots \\ \tilde{K}_1(x_m,y) & \tilde{K}_1(x_m,x_1) & \cdots & \tilde{K}_1(x_m,x_m) \end{vmatrix} dx_1 dx_2 \cdots dx_m .$$

Consider the discriminant of $\tilde{K}_1$

$$D(\mu) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} D_n(\ell)\ \mu^n$$

$D(\mu)$ is an entire transcendental function in μ and ℓ . The zeros of $D(\mu)$ are the reciprocals of the eigenvalues of (4.1) i.e. $\frac{1}{\tilde{\lambda}_i(\ell)} = \mu_i(\ell)$, where $\mu_i(\ell)$ are the zeros of $D(\mu)$ for any given ℓ .

For any fixed ℓ_o, if $\mu_i(\ell_o)$ is a simple zero of $D(\mu)$, we can find a small enough circle in the μ plane around $\mu_i(\ell_o)$ since the spectrum is discrete, such that

$$\frac{1}{2\pi i}\oint \frac{1}{D(\mu)}\, d\mu = a_i(\ell_o)$$

and

$$\frac{1}{2\pi i}\oint \frac{\mu}{D(\mu)}\, d\mu = a_i(\ell_o)\,\mu_i(\ell_o)$$

where $a_i(\ell_o)$ is the residue of $\frac{1}{D(\mu)}$ at $\mu_i(\ell_o)$.

We observe that since $\frac{1}{D(\mu)}$ and $\frac{\mu}{D(\mu)}$ are analytic functions of ℓ, if we can find a neighborhood $\varepsilon(\ell_o)$ and a corresponding neighborhood $\varepsilon'\left[\mu_i(\ell_o)\right]$ such that for any ℓ in $\varepsilon(\ell_o)$ all $\mu_j(\ell)$ $j \neq i$ are excluded from $\varepsilon'\left[\mu_i(\ell_o)\right]$ then $a_i(\ell)$ and $a_i(\ell)\mu_i(\ell)$ are analytic functions of in $\varepsilon(\ell_o)$, hence $\mu_i(\ell)$ is an analytic function in $\varepsilon(\ell_o)$.

For the kernel $\tilde{K}_1$ we have

$$D_1(\ell) = 2\gamma\theta_o$$

$$D_2(\ell) = -\gamma^2\left[(k\ell)^2\theta_o\theta_2\,\frac{16}{3} + (k\ell)^4\,(2\theta_o\theta_4 + \theta_2^2)\,\frac{64}{15} + \cdots\cdots\right]$$

$$D_3(\ell) = \gamma^3\left[(k\ell)^4\,\theta_o\theta_2^2\,\frac{64}{15} + \cdots\cdots\right].$$

In general $D_n(\ell)$ are even power series in $k\ell$ where the lowest order term in $D_n(\ell)$ has the power $2n-2$.

We are now able to demonstrate the analyticity of $\tilde{\lambda}_1(\ell)$ in the neighborhood of $\ell = 0$.

If in $D(\mu)$ we set $\ell = 0$ we get $D(\mu) = 1 - 2\gamma\theta_o\mu$. This has only the zero $\mu_1 = \frac{1}{2\gamma\theta_o}$. All other μ_i become infinite. We now show that there must be a neighborhood of $\ell = 0$ in which $\mu_i(\ell) - \mu_i(\ell) > \delta > 0$, δ a constant for all $i \neq 1$.

We note that

1) $\mu_i(\ell)$ are continuous for $\ell \neq 0$.

2) $\mu_1(\ell) = \min\left\{ \mu(\ell) \right\}$

3)
$$\mu_2(\ell) = \left\{ \begin{array}{ll} \min\left\{ \mu(\ell) \,|\, \mu(\ell) > \mu_1(\ell) \right\}, & \text{for those } \ell \text{ where } \mu_1(\ell) \text{ is simple} \\ \mu_1(\ell) & \text{, for } \ell \text{ where } \mu_1(\ell) \text{ is multiple} \end{array} \right\} .$$

For $i \neq 1$ all $\mu_i(\ell)$ and particularly $\mu_2(\ell)$ tend to infinity as $\ell \to 0$, and since

$$\lim_{\ell \to 0} \mu_1(\ell) = \frac{1}{2\theta_o\gamma}$$

there is a neighborhood of $\ell = 0$ where $\mu_i(\ell) - \mu_1(\ell) > \delta > 0$. δ, a constant.

Therefore $\mu_1(\ell)$ is an analytic function of ℓ in this neighborhood.

From the knowledge of the regular analyticity of $\mu_1(\ell)$ in the neighborhood of zero, we find that

$$\tilde{\lambda}_1(\ell) = \frac{1}{\mu_1(\ell)} = 2\theta_o\gamma + \frac{4}{3}\theta_2 k^2 \ell^2 + O(\ell^4) .$$

We observe that $\lambda_1(\ell)$ is simple around $\ell = 0$ and we now investigate the

behavior of its einfunction $f_1(x)$ around $\ell = 0$.

Let $$D(x,y;\mu) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} D_m(x,y)\, \mu^m$$

where $D_o(x,y) = \tilde{K}_1(x,y)$.

$D(x,y;\mu)$ is regular analytic for finite values of x,y,ℓ and μ. We now consider the resolvent of K_1 ,

$$\mathbf{K}(x,y;\mu) = \frac{D(x,y;\mu)}{D(\mu)}$$

which is a meromorphic function with poles at the zeros of $D(\mu)$. Another representation of the resolvent is

$$\mathbf{K}(x,y;\mu) = \tilde{K}_1(x,y) + \mu \sum_{i=1}^{\infty} \frac{f_1(x) f_i(y)}{\mu_i(\mu_i - \mu)}$$

where the f_i are the normalized eigenfunctions of the μ_i.

If we take a small enough circle in the μ plane around $\mu = \frac{1}{2\gamma\theta_o}$

$$\frac{1}{2\pi i} \oint \mathbf{K}(x,y;\mu)d\mu = f_1(x)\, f_1(y)\, \frac{1}{2\pi i} \oint \frac{\mu}{\mu_1(\mu_1 - \mu)}\, d\mu$$

$$= f_1(x)\, f_1(y)$$

since the left hand side is analytic in ℓ around $\ell = 0$ we get the result $f_1(x)$ is an analytic function, regular in a neighborhood of $\ell = 0$.

The first two terms of the expansion of $f_1(x)$ are

$$f_1(x) = \frac{1}{\sqrt{2}} \left[1 + \frac{\theta_2}{\theta_o} k^2 \ell^2 (x^2 - \tfrac{1}{3})\right] + O(\ell^4) \quad .$$

The second eigenvalue and eigenfunction.

Let $K_1^{(2)} = K_1 - \lambda_1 f_1(x) f_1(y)$ and form the equation

$$\lambda^{(2)} f(y) = \int K_1^{(2)}(x,y)\, f(x)\, dx \qquad . \tag{4.2}$$

Let $\frac{\lambda^{(2)}}{k^2 \ell^3} = \tilde{\lambda}^{(2)}$ and $\frac{1}{k^2 \ell^3} K_1^{(2)} = \tilde{K}_1^{(2)}$ we then get

$$\tilde{\lambda}^{(2)} f(y) = \int \tilde{K}_1^{(2)} f(x)\, dx \tag{4.3}$$

$$\tilde{K}_1^2 = \Theta_2 \left[-2xy + \frac{2}{3}(\gamma - 1) \right] + O(\ell^2)$$

If we form the discriminant $D^{(2)}(\mu)$ and set $\ell = 0$, we get

$$\mu_1^{(2)}(0) = -\frac{1}{\frac{4}{3}\Theta_2} \qquad .$$

By the same argument and reasoning as for $\tilde{\lambda}_1$ we get $\tilde{\lambda}_1^{(2)}(\ell)$ is simple and analytic around $\ell = 0$, and

$$\tilde{\lambda}_1^{(2)}(\ell) = -\frac{4}{3}\Theta_2 + O(\ell^2)$$

Similarly we establish $f_2(x)$ the eigenfunction of $\lambda_1^{(2)}$ is analytic around $\ell = 0$ and

$$f_2(x) = \sqrt{\frac{3}{2}}\, x + O(\ell^3) \qquad .$$

$$\lambda_1(\ell) = \ell\tilde{\lambda}_1(\ell) = 2\Theta_0\gamma\ell + \frac{4}{3}\Theta_2 k^2\ell^3 + O(\ell^5)$$

and

$$\lambda_2(\ell) = k^2\ell^3\,\lambda_1^{(2)}(\ell) = -\frac{4}{3}\Theta_2 k^2\ell^3 + O(\ell^5) \quad .$$

We note that further computation gives for the first normalized eigenfunction of A,

$$\phi_1(x) = \frac{1}{\Theta_2\sqrt{2}}\,\frac{J_{3/2}(kx)}{(kx)^{3/2}} + O(\ell^2) \quad .$$

5. Direct methods.

If we expand the current-distribution $f(s) = \alpha(s) + i\beta(s)$ in a Fourier series in $-\ell \leq s \leq \ell$, the radiated energy is a quadratic form in the Fourier coefficients of $\alpha(s)$ and $\beta(s)$. First, we shall compute the coefficients of this quadratic form. Next, we shall obtain a lower bound for the maximum of this quadratic form (under the side-condition of a constant heat loss) by taking all but a finite number of the Fourier coefficients of f(s) equal to zero. In particular, the coefficients of the squares of the Fourier-coefficients in our quadratic form provide lower bounds for the maximum of the radiated energy, i.e. for the largest eigenvalue λ_1 of the integral equation considered in the last section.

Let $f(s) = \alpha(s) + i\beta(s)$,

$$\alpha = \sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi}{\ell} s\right) + A_n \sin\left(\frac{n\pi}{\ell} s\right) ,$$

and

$$\beta = \sum_{n=0}^{\infty} b_n \cos\left(\frac{n\pi}{\ell} s\right) + B_n \sin\left(\frac{n\pi}{\ell} s\right) .$$

The radiated energy is given by

$$\mathcal{E} = \frac{2\pi k^3}{\omega} \int_0^{\pi} \sin^3\theta \left\{ \left(\int_{-\ell}^{\ell} \alpha(s)\cos(sk\cos\theta)ds + \int_{-\ell}^{\ell} \beta(s)\sin(sk\cos\theta)ds \right)^2 + \left(\alpha(s)\sin(sk\cos\theta)ds - \int_{-\ell}^{\ell} \beta(s)\cos(sk\cos\theta)ds \right)^2 \right\} d\theta$$

Let $C_n(\theta) = \int_{-\ell}^{\ell} \cos\left(\frac{n\pi}{\ell} s\right) \cdot \cos(sk\cos\theta)\, ds$

and $S_n(\theta) = \int_{-\ell}^{\ell} \sin\left(\frac{n\pi}{\ell} s\right) \cdot \sin(sk\cos\theta)\, ds$

Then

$$\mathcal{E} = \frac{2\pi k^3}{\omega} \int_0^{\pi} \sin^3\theta \left\{ \left(\sum_{n=0}^{\infty} a_n C_n(\theta) + B_n S_n(\theta) \right)^2 + \left(\sum_{n=0}^{\infty} A_n S_n(\theta) - b_n C_n(\theta) \right)^2 \right\} d\theta$$

$$= \frac{2\pi k^3}{\omega} \int_0^{\pi} \left\{ \sum_{n,m=0}^{\infty} \begin{array}{c} a_n^2 C_n^2(\theta) + B_n^2 S_n^2(\theta) + 2a_n a_m C_n(\theta) C_m(\theta) \\ +2B_n B_m S_n(\theta) S_m(\theta) + 2a_n B_m C_n(\theta) S_m(\theta) \\ +a_n B_n C_n(\theta) S_n(\theta) \\ A_n^2 S_n^2(\theta) + b_n C_n^2(\theta) + 2b_n b_m C_n(\theta) C_m(\theta) \\ +2A_n A_m S_n(\theta) S_m(\theta) - 2b_n A_m C_n(\theta) S_m(\theta) \\ -b_n A_n C_n(\theta)\, S_n(\theta) \end{array} \right\} d\theta$$

$$S_n(\theta) = \frac{(-1)^n 2n\pi \ell \sin(k\ell\cos\theta)}{(k\ell\cos\theta)^2 - (n\pi)^2}$$

$$C_n(\theta) = \frac{(-1)^n 2\ell(k\ell\cos\theta)\sin(k\ell\cos\theta)}{(k\ell\cos\theta)^2 - (n\pi)^2}$$

$$S_n^2(\theta) = \frac{4(n\pi\ell)^2\sin^2(k\ell\cos\theta)}{\left[(k\ell\cos\theta)^2-(n\pi)^2\right]^2}$$

$$C_n^2(\theta) = \frac{4\ell^2(k\ell\cos\theta)^2\sin^2(k\ell\cos\theta)}{\left[(k\ell\cos\theta)^2-(n\pi)^2\right]^2}$$

$$S_n(\theta)C_m(\theta) = \frac{(-1)^{n+m}4n\pi\ell^2(\ell\cos\theta)\sin^2(k\ell\cos\theta)}{\left[(k\ell\cos\theta)^2-(n\pi)^2\right]\left[(k\ell\cos\theta)^2-(m\pi)^2\right]}$$

$$S_n(\theta)S_m(\theta) = \frac{(-1)^{n+m}4nm(\pi\ell)^2\sin^2(k\ell\cos\theta)}{\left[(k\ell\cos\theta)^2-(n\pi)^2\right]\left[(k\ell\cos\theta)^2-(m\pi)^2\right]}$$

$$C_n(\theta)C_m(\theta) = \frac{(-1)^{n+m}4\ell^2(k\ell\cos\theta)^2\sin^2(k\ell\cos\theta)}{\left[(k\ell\cos\theta)^2-(n\pi)^2\right]\left[(k\ell\cos\theta)^2-(m\pi)^2\right]}$$

If in the following integrals we substitute for the $C_i(\theta)C_j(\theta)$, $S_i(\theta)S_j(\theta)$, $C_i(\theta)S_j(\theta)$ the above evaluations and make the change of variable $t = k\ell\cos\theta$, then we have

$$T(C)_{n,m} = \int_0^\pi \sin^3\theta C_n^2(\theta)d\theta = \frac{1}{(k\ell)^3}\int_{-k\ell}^{k\ell}\left[(k\ell)^2-t^2\right]\frac{4\ell^2t^2\sin^2(t)}{\left[t^2-(n\pi)^2\right]^2}\,dt$$

$$T(S)_{n,m} = \int_0^\pi \sin^3\theta S_n^2(\theta)d\theta = \frac{1}{(k\ell)^3}\int_{-k\ell}^{k\ell}\left[(k\ell)^2-t^2\right]\frac{4(n\pi\ell)^2\sin^2t}{\left[t^2-(n\pi)^2\right]^2}\,dt$$

$$T(C)_{n,m} = \int_0^{\pi} \sin^3\theta\, C_n(\theta) C_n(\theta)\, d\theta = \frac{(-1)^{n+m}}{(k\ell)^3} \int_{-k\ell}^{k\ell} \left[(k\ell)^2 - t^2\right] \frac{4\ell^2 t^2 \sin^2 t}{\left[t^2-(n\pi)^2\right]\left[t^2-(m\pi)^2\right]}\, dt$$

$$T(S)_{n,m} = \int \sin^3\theta\, S_n(\theta) S_m(\theta)\, d\theta = \frac{(-1)^{n+m}}{(k\ell)^3} \int_{-k\ell}^{k\ell} \left[(k\ell)^2 - t^2\right] \frac{4nm(\pi\ell)^2 \sin^2 t}{\left[t^2-(n\pi)^2\right]\left[t^2-(m\pi)^2\right]}\, dt$$

$$\int_0^{\pi} \sin^3\theta\, S_n(\theta)\, C_m(\theta)\, d\theta = 0$$

We therefore have

$$\mathcal{E} = \frac{2\pi k^3}{\omega} \left\{ \sum_{n,m=0}^{\infty} T(C)_{n,m} a_n a_m + \sum_{n,m=0}^{\infty} T(C)_{n,m} b_n b_m + \sum_{n,m=0}^{\infty} T(S)_{n,m} A_n A_m + \sum_{n,m=0}^{\infty} T(S)_{n,m} B_n B_m \right\}$$

By long, but elementary, computations, for fixed n, m we get as $\ell \to \infty$,

$$T(S)_{n,n} = \frac{2\pi}{k}\ell + o(\ell^{\varepsilon-1})$$

$$T(C)_{n,n} = \frac{2\pi}{k}\ell - \frac{8}{k^2} + o(\ell^{\varepsilon-1})$$

$$T(S)_{n,m} = o(\ell^{\varepsilon-1})$$

$$T(C)_{n,m} = (-1)^{n+m+1} \frac{4}{k^2} + o(\ell^{\varepsilon-1}) \qquad \varepsilon > 0 \ .$$

These results imply the following statement: For large values of ℓ, a lower bound for the maximum of the radiated energy is given by

$$\frac{(2\pi k)^2}{\omega} + o(\ell^{\varepsilon-2})$$

if we take only a finite number of $\{A_n\}$ and $\{B_n\}$ to be $\neq 0$ and all $\{a_n\}$ and $\{b_n\}$ to be $= 0$.

6. Limit and upper bound for the radiated energy.

The largest eigenvalue of our integral equation (3.6) is the maximum for the quotient $\frac{\mathcal{E}}{\mathcal{H}}$. If in our expression for $\mathcal{E}$ on page 13, we assume for a fixed ℓ, $f(s) = \alpha_1(s) + i\beta_1(s)$ where $\alpha_1(s)$ and $\beta_1(s)$ are eigenfunctions of $\lambda_1(\ell)$, then as on page 20 we get

$$\lambda_1(\ell) = \frac{2\pi k^2}{\omega} \int_{-k}^{k} \left(1 - \frac{t^2}{k^2}\right) g(t)\, \overline{g}(t)\, dt$$

if we assume $\mathcal{H}(f) = \int_{-\infty}^{\infty} f(s)\, \overline{f}(s)\, ds = \int_{-\ell}^{\ell} f(s)\, \overline{f}(s)\, ds = 1$

and let $$g(t) = \begin{cases} G(t) = \int_{-\infty}^{\infty} e^{ist} f(s)\, ds \quad , & -k \leq t \leq k \\ 0 & |t| > k \end{cases} .$$

Since $g(t)\, \overline{g(t)} \geq 0$ and $1 - \frac{t^2}{k^2} \leq 1$, we get

$$\lambda_1(\ell) \leq \frac{2\pi k^2}{\omega} \int_{-k}^{k} g(t)\, \overline{g(t)}\, dt \leq \frac{2\pi k^2}{\omega} \int_{\infty}^{\infty} G(t)\, \overline{G(t)}\, dt \quad .$$

Since $\int_{-\infty}^{\infty} G(t)\,\overline{G(t)}\,dt = 2\pi \int_{-\infty}^{\infty} f(s)\,\overline{f(s)}\,ds = 2\pi$

we get as a result for all $0 < \ell < \infty$

$$\lambda_1(\ell) \leq \frac{(2\pi k)^2}{\omega} \quad .$$

Combining this with the result of the last section we have

$$\frac{(2\pi k)^2}{\omega} + o(\ell^{\varepsilon-2}) \leq \lambda_1(\ell) \leq \frac{(2\pi k)^2}{\omega}$$

hence

$$\lim_{\ell \to \infty} \lambda_1(\ell) = \frac{(2\pi k)^2}{\omega}$$

This can be summarized in the following statement. The maximum radiated energy (assuming the heat losses = 1) is a non-decreasing function of ℓ which approaches $\frac{(2\pi k)^2}{\omega}$ as $\ell \to \infty$.

7. Integral equations for the second problem.

The energy radiated to upper half-space, $\mathcal{E}^*$, is a functional of the current distribution $f(s) = \alpha(s) + i\beta(s)$. If we write

$$\mathcal{E}^* = \gamma^* \int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \left\{ \begin{array}{l} \dfrac{J_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}} \left(\alpha(u)\alpha(v) + \beta(u)\beta(v)\right) \\ + \dfrac{H_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}} \left(\alpha(u)\beta(v) - \alpha(v)\beta(u)\right) \end{array} \right\} dudv$$

where $\gamma^* = \frac{\gamma}{2}$, the problem of maximizing $\frac{\mathcal{E}^*(f)}{\mathcal{H}(f)}$ over the class of functions $\alpha(s) + i\beta(s) = f(s) \in L^2(-\ell,\ell)$ (s real), is the same as that of maximizing $\mathcal{E}^*(f)$ subject to the condition $\mathcal{H}(f)$ constant, over the same class of functions.

Setting the first variation = 0 gives equations

$$(7.1) \quad \gamma^* \int_{-\ell}^{\ell} \Big[k(u,v)\alpha(u) - L(u,v)\beta(u)\Big]\, du = \lambda^*\alpha(v)$$

$$(7.2) \quad \gamma^* \int_{-\ell}^{\ell} \Big[L(u,v)\alpha(u) + k(u,v)\beta(u)\Big]\, du = \lambda^*\beta(v)$$

where $k(u,v) = \dfrac{J_{3/2}\Big[k(u-v)\Big]}{\Big[k(u-v)\Big]^{3/2}}$, $L(u,v) = \dfrac{\mathbf{H}_{3/2}\Big[k(u-v)\Big]}{\Big[k(u-v)\Big]^{3/2}}$.

Let $k^*(u,v) = \gamma^*\Big[k(u,v) + iL(u,v)\Big]$ and $\phi(s) = \alpha(s) + i\beta(s)$, then (7.1) and (7.2) can be combined into

$$(7.3) \quad \int_{-\ell}^{\ell} k^*(u,v)\, \phi(u)\, du = \lambda^*\phi(v) \quad .$$

If in (7.3) we let $u = \ell x$ and $v = \ell y$, we get

$$(7.4) \quad \int_{-1}^{1} k^*(\ell x, \ell y)\, f(x)\, dx = \lambda^* f(y)$$

where $f(x) = \phi(\ell x)$ and $f(y) = \phi(\ell y)$.

We note that

$$\int_{-\infty}^{\infty} e^{ivs}\, k^*(u,v)\, dv = \left\{ \begin{array}{ll} \sqrt{\frac{\pi}{2}}\ \frac{\gamma}{k^3}\, e^{ius}\, (k^2 - s^2) , & 0 \le s \le k \\ 0 , & \text{elsewhere} \end{array} \right\} .$$

Letting

$$g(s) = \int_{-\infty}^{\infty} e^{irs}\, \phi(v)$$

and carrying through the same procedure as on pp. 15, 16 and 17, we get

$$(7.5) \quad \lambda^* h(y) = \frac{\gamma}{\sqrt{2\pi}} \int_0^1 \frac{\sin\left[k\ell(x-y)\right]}{k(x-y)} \sqrt{1-y^2}\ \sqrt{1-x^2}\ h(x)\, dx$$

and

$$(7.6) \quad \lambda^* h(y) = \int_0^1 \frac{\gamma}{2}\ell\, \frac{J_{1/2}\left[k\ell(x-y)\right]}{\left[k\ell(x-y)\right]^{1/2}} \sqrt{1-y^2}\ \sqrt{1-x^2}\ h(x)\, dx \quad .$$

We will denote the kernels of (7.3), (7.4) and (7.6) by k^*, k_1^* and k_2^* respectively. The set of eigenvalues of these equations is the same. k^* is analytic in all of its arguments for $0 < \ell < \infty$ and Hermitian, therefore for $0 < \ell < \infty$ the equations have a real discrete spectrum.

We now show that k^*, k_1^*, k_2^* are positive definite. The proof of definiteness follows the same procedure as on p. 18. We note that

$$\frac{J_{1/2}\left[k\ell(x-y)\right]}{\left[k\ell(x-y)\right]^{1/2}} = \sum_{n=0}^{\infty} \frac{J_{1/2+n}(k\ell x)}{(k\ell x)^{1/2}} \cdot \frac{J_{1/2+n}(k\ell y)}{(k\ell y)^{1/2}} \quad .$$

Therefore, for k_2^*, for any admissible h(s)

$$\int_0^1 \int_0^1 \frac{\gamma\ell}{2} \frac{J_{1/2}\left[k\ell(x-y)\right]}{\left[k\ell(x-y)\right]^{1/2}} \sqrt{1-x^2}\,\sqrt{1-t^2}\; h(x)\; h(y)\; dx\, dy$$

$$= \frac{\gamma\ell}{2} \sum_{n=0}^{\infty} \int_0^1 \frac{J_{1/2+n}(k\ell x)}{(k\ell x)^{1/2}} \sqrt{1-x^2}\; h(x)dx \int_0^1 \frac{J_{1/2+n}(k\ell y)}{(k\ell y)^{1/2}} \sqrt{1-y^2}\; h(y)dy$$

$$= \frac{\gamma\ell}{2} \sum_{n=0}^{\infty} \left[\int_0^1 \frac{J_{1/2+n}(k\ell s)}{(k\ell s)^{1/2}} \sqrt{1-s^2}\; h(s)\; ds \right]^2 \qquad \geq 0 \quad .$$

Hence k_2^* is positive and therefore positive definite. Therefore the spectrum of k_2^* for $0 < \ell < \infty$ is positive. So that the k^* and k_1^* are also positive definite kernels.

By the same procedure as on p. 19, it can be shown that $\lambda_1^*(\ell)$, the largest eigenvalue is a non-decreasing function of .

8. Behavior of $\lambda_1^*(\ell)$, the largest eigenvalue of the second problem, for small ℓ.

Rewrite (7.4) in the form

$$(8.1)\quad \tilde{\lambda}_1^*\, f(y) = \int_{-1}^{1} \tilde{k}_1^*(x,y)\; f(x)dx$$

where $\tilde{\lambda}_1^* = \dfrac{\lambda_1^*}{\ell}$ and $\tilde{k}_1^*(x,y) = \gamma^* \left[\dfrac{J_{3/2}\left[k\ell(x-y)\right]}{\left[k\ell(x-y)\right]^{3/2}} + i\, \dfrac{\mathbf{H}_{3/2}\left[k\ell(x-y)\right]}{\left[k\ell(x-y)\right]^{3/2}} \right]$.

We note that

$$\frac{J_{3/2}\left[k\ell(x-y)\right]}{\left[k\ell(x-y)\right]^{3/2}} = \sum_{n=0}^{\infty} \Theta_{2n} \left[k\ell(x-y)\right]^{2n}$$

$$\text{where } \Theta_{2n} = (-1)^n \left[\frac{1}{(2n+2)!} - \frac{1}{(2n+3)!} \right] ,$$

and

$$\frac{\mathbf{H}_{3/2}\left[k\ell(x-y)\right]}{\left[k\ell(x-y)\right]^{3/2}} = \sum_{n=0}^{\infty} \Theta_{2n+1} \left[k\ell(x-y)\right]^{2n+1}$$

$$\text{where } \Theta_{2n+1} = (-1)^n \left[\frac{1}{(2n+3)!} - \frac{1}{(2n+4)!} \right] .$$

$\tilde{k}_1^*(x,y)$ is an entire function of $k\ell$ for bounded x and y.

If we follow the procedure of part 4 and form the discriminant and resolvent of (8.1) and reasoning in the same way as there, we establish

1) that $\lambda_1^*(\ell)$ is simple and analytic around $\ell = 0$

and

2) $f_1(x)$, the eigenfunction of $\lambda_1^*(\ell)$ is an analytic function of ℓ around $\ell = 0$.

We can then compute

$$\lambda_1^*(\ell) = \gamma\left[\Theta_o \ell + \left(\frac{2}{3}\Theta_2 + \frac{1}{3}\frac{\Theta_1}{\Theta_o}\right) k^2 \ell^3 + O(\ell^4)\right]$$

and

$$f_1^*(x) = \frac{1}{\Theta_o\sqrt{2}} \left\{ \begin{array}{l} \Theta_o \quad k^2\ell^2(\Theta_2 x^2 - \frac{\Theta_2}{3} - \frac{1}{6}\frac{\Theta_1^2}{\Theta_o}) + O(\ell^4) \\ \\ + i\left[\Theta_1 k\ell x + O(\ell^3)\right] \end{array} \right\} ,$$

the normalized eigenfunction of $\lambda_1^*(\ell)$.

We notice that $\lambda_1^*(\ell) > \frac{\lambda_1(\ell)}{2}$ around $\ell = 0$. In the next part we establish that this holds for $0 < \ell < \infty$.

9. Bounds and limits for the radiated energy in the second problem.

1) $\lambda_1(\ell) > \lambda_1^*(\ell)$ for $0 < \ell < \infty$.

Let $\phi_1(s)$ and $\phi_1^*(s)$ be eigenfunctions of λ_1 in (3.6) and λ_1^* in (7.3) respectively.

Then

$$(9.1)\quad \lambda_1(\ell) = \frac{2\pi k^3}{\omega} \int_0^{\pi} \sin^3\theta \left[\int_{-\ell}^{\ell} e^{-iks\cos\theta} \phi_1(s)ds \int_{-\ell}^{\ell} e^{-iks\cos\theta} \phi_1(s)ds \right] d\theta$$

and

$$(9.2)\quad \lambda_1^*(\ell) = \frac{2\pi k^3}{\omega} \int_{\frac{\pi}{2}}^{\pi} \sin^3\theta \left[\int_{-\ell}^{\ell} e^{-iks\cos\theta} \phi_1^*(s)ds \int_{-\ell}^{\ell} e^{-iks\cos\theta} \phi_1^*(s)ds \right] d\theta \quad .$$

If in (9.1) we replace $\phi_1(s)$ by $\phi_1^*(s)$, and let $k\cos\theta = t$, we get

$$(9.3)\quad \lambda_1(\ell) \geq \frac{2\pi k^2}{\omega} \int_{-k}^{k} \left(1 - \frac{t^2}{k^2}\right) \left[\int_{-\ell}^{\ell} e^{its} \phi_1^*(s)ds \overline{\int_{-\ell}^{\ell} e^{-its} \phi_1^*(s)ds} \right] d\theta$$

similarly for (9.2) we get

$$(9.4)\quad \lambda_1^*(\ell) = \frac{2\pi k^2}{\omega} \int_{-k}^{0} \left(1 - \frac{t^2}{k^2}\right) \left[\int_{-\ell}^{\ell} e^{-its} \phi_1^*(s)ds \overline{\int_{-\ell}^{\ell} e^{-its} \phi_1^*(s)ds} \right] d\theta$$

since $\phi_1^*(s) \equiv 0 \quad |s| > \ell$ we get

$$(9.5)\quad G^*(t) = \int_{-\infty}^{\infty} e^{-its}\phi_1^*(s)ds = \int_{-\ell}^{\ell} e^{-its}\phi_1^*(s)ds$$

where $G^*(t)$ is analytic in t.

We rewrite (9.3) using (9.4) and (9.5).

$$(9.6)\quad \lambda_1(\ell) \geq \lambda_1^*(\ell) + \frac{2\pi k^2}{\omega}\int_{-k}^{0} \left(\frac{1-2^2}{k^2}\right) G^*(t)\,\overline{G^*(t)}\,dt \quad .$$

Since $|G^*(t)|^2$ can't vanish identically for $-k < t < 0$ the integral on the right is positive, therefore

$$\lambda_1(\ell) > \lambda_1^*(\ell) \quad .$$

$$2)\quad \lambda_1^*(\ell) > \frac{\lambda_1(\ell)}{2} \quad \text{for } 0 < \ell < \infty \ .$$

Let $h_1(s)$ be an even or odd normalized eigenfunction of $\lambda_1(\ell)$ (there exists at least one) in equation (3.11).

Then

$$\lambda_1(\ell) = \frac{\gamma}{\sqrt{2\pi}} \int_{-1}^{1}\int_{-1}^{1} \frac{\sin\left[k\ell(x-y)\right]}{\left[k\ell(x-y)\right]} \sqrt{1-x^2}\ \sqrt{1-y^2}\ h_1(x)\,\overline{h_1(y)}\,dxdy \ .$$

Let

$$P(x-y) = \frac{\gamma\ell}{\sqrt{2\pi}} \sqrt{1-x^2}\ \sqrt{1-y^2}\ \frac{\sin\left[k\ell(x-y)\right]}{k\ell(x-y)}$$

and

$$P(x+y) = \frac{\gamma \ell}{\sqrt{2\pi}} \sqrt{1-x^2}\,\sqrt{1-y^2}\ \frac{\sin\left[k\ell\,(x+y)\right]}{k\ell\,(x+y)} \quad .$$

Then

$$\frac{\lambda_1(\ell)}{2} = \int_0^1 \int_0^1 \left[P(x-y) + P(x+y)\right]\, h_1(x)\, \overline{h_1(y)}\, dxdy$$

when $h_1(s)$ is even and

$$\frac{\lambda_1(\ell)}{2} = \int_0^1 \int_0^1 \left[P(x-y) - P(x+y)\right]\, h_1(x)\, \overline{h_1(y)}\, dxdy$$

when $h_1(s)$ is odd.

Since

$$\frac{\sin\left[k\ell(x-y)\right]}{k\ell(x-y)} = \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{J_{1/2+n}(k\ell x)}{(k\ell x)^{1/2}}\ \frac{J_{1/2+n}(k\ell y)}{(k\ell y)^{1/2}} \quad ,$$

$$\frac{\sin\left[k\ell(x+y)\right]}{k\ell(x+y)} = \sqrt{\frac{\pi}{2}} \sum_{n=0}^{\infty} \frac{J_{1/2+n}(k\ell x)}{(k\ell x)^{1/2}}\ \frac{J_{1/2+n}(-k\ell y)}{(-k\ell y)^{1/2}} \quad ,$$

and $\dfrac{J_{n+1/2}(\zeta)}{(\zeta)^{1/2}}$ is an odd function of ζ for n odd and an even function for n even, we get

$$P(x-y) - P(x+y) = \ell\gamma \sqrt{1-x^2}\,\sqrt{1-y^2} \sum_{n=0}^{\infty} \frac{J_{2n+1+1/2}(k\ell x)}{(k\ell x)^{1/2}}\ \frac{J_{2n+1+1/2}(k\ell y)}{(k\ell y)^{1/2}}$$

and

$$P(x-y) + P(x+y) = \ell\gamma \sqrt{1-x^2}\,\sqrt{1-y^2} \sum_{n=0}^{\infty} \frac{J_{2n+1/2}(k\ell x)}{(k\ell x)^{1/2}}\ \frac{J_{2n+1/2}(k\ell y)}{(k\ell y)^{1/2}} \quad .$$

Therefore

$$\int_0^1 \int_0^1 \left[P(x-y) - P(x+y)\right] h_1(x)\, \overline{h_1(y)}\, dxdy$$

$$= \ell\gamma \sum_{n=0}^{\infty} \left| \int_0^1 \frac{J_{2n+1+1/2}(k\ell s)}{(k\ell s)^{1/2}} \sqrt{1-s^2}\, h_1(s)ds \right|^2 \geq 0$$

and

$$\int_0^1 \int_0^1 \left[P(x-y) + P(x+y)\right] h_1(x)\, \overline{h_1(y)}\, dxdy$$

$$= \ell\gamma \sum_{n=0}^{\infty} \left| \int_0^1 \frac{J_{2n+1/2}(k\ell s)}{(k\ell s)^{1/2}} \sqrt{1-s^2}\, h_1(s)ds \right|^2 \geq 0 \quad .$$

Since $\left\{\dfrac{J_{2n+1/2}(\zeta)}{\zeta^{1/2}}\right\}$ and $\left\{\dfrac{J_{2n+1+1/2}(\zeta)}{\zeta^{1/2}}\right\}$ are

complete in any finite interval (0, q), the strict inequalities hold. It then follows that

$$\int_0^1 \int_0^1 \left[P(x-y) + P(x+y)\right] h_1(x)\overline{h_1(y)}\, dxdy < 2 \int_0^1 \int_0^1 P(x-y)\, h_1(x)\overline{h_1(y)}\, dxdy$$

and

$$\int_0^1 \int_0^1 \left[P(x-y) - P(x+y)\right] h_1(x)\overline{h_1(y)}\, dxdy < 2 \int_0^1 \int_0^1 P(x-y)\, h_1(x)\overline{h_1(y)}\, dxdy \quad .$$

Therefore

$$2 \int_0^1 \int_0^1 P(x-y)\, h_1(x)\, \overline{h_1(y)}\, dxdy > \frac{\lambda_1(\ell)}{2} \quad .$$

Since $\int_0^1 \left[\sqrt{2}\, h_1(s)\right]^2 ds = 1$, it follows from equation (7.5) that,

$$\lambda_1^*(\ell) \geq \frac{2\gamma\ell}{\sqrt{2\pi}} \int_0^1 \int_0^1 \frac{\sin[k\ell(x-y)]}{k\ell(x-y)} \sqrt{1-x^2}\sqrt{1-y^2}\, h_1(x)\, h_1(y)\, dxdy$$

$$= 2\int_0^1 \int_0^1 P(x-y)\, h_1(x)\, \overline{h_1(y)}\, dxdy \quad .$$

We therefore get

$$\lambda_1^*(\ell) > \frac{\lambda_1(\ell)}{2} \quad .$$

3) Limit and bounds for $\lambda_1^*(\ell)$ at infinity

Consider $f(x)$ such that $f(x) \equiv 0$ for $0 < x$ and $x > 1$, x continuous of bounded variation and $\int_0^1 f(x)\overline{f}(x)\, dx = 1$.

By Dirichlet's theorem it follows that

$$\lim_{\ell \to \infty} \int^{\infty} \frac{\sin k\ell(x-y)}{x-y} \sqrt{1-x^2}\, f(x)\, dx = \pi f(y) \sqrt{1-y}$$

and therefore

$$(9.7)\quad \lim_{\ell \to \infty} \int_0^1 \int_0^1 \frac{\sin\left[k\ell(x-y)\right]}{x-y} \sqrt{1-x^2}\sqrt{1-y^2}\, f(x)\, \overline{f(y)}\, dx\, dy$$

$$= \pi \int_0^1 (1-y^2)\, f(y)\, \overline{f(y)}\, dy \leq \pi$$

for all functions in the class described above.

$$\lim_{\ell \to \infty} \lambda_1^*(\ell) \geq \lim_{\ell \to \infty} \frac{\gamma}{\sqrt{2\pi}} \frac{1}{k} \int_0^1 \int_0^1 \frac{\sin[k\ell(x-y)]}{x-y} \sqrt{1-x^2}\, \sqrt{1-y^2}\, f(x)\overline{f(y)}\, dxdy$$

$$= \frac{4\pi^2 k^2}{\omega} \int_0^1 (1-y^2)\, f(y)\, \overline{f(y)}\, dy$$

for all functions in the class described above.

Let

$$f_\varepsilon(s) = \begin{cases} \dfrac{1}{\sqrt{\varepsilon}} & , \quad 0 \leq s \leq \varepsilon \\ 0 & , \quad \text{elsewhere} \end{cases} .$$

If we insert $f_\varepsilon(s)$ in (9.9) and let $\ell \to \infty$, then by (9.7) we get

$$\lim_{\ell \to \infty} \lambda_1^*(\ell) \geq \frac{4\pi^2 k^2}{\omega} \left(1 - \frac{\varepsilon}{3}\right)$$

for any $\varepsilon > 0$. If we now let $\varepsilon \to 0$, we get

$$\lim_{\ell \to \infty} \lambda_1^*(\ell) \geq \frac{4\pi^2 k^2}{\omega} .$$

By the same reasoning as in Section 6, (p. 31 and 32) we establish that

$$\lambda_1^*(\ell) \leq \frac{4\pi^2 k^2}{\omega}$$

for all ℓ. Hence

$$\lim_{\ell \to \infty} \lambda_1^*(\ell) \leq \frac{4\pi^2 k^2}{\omega} .$$

We therefore get the result

$$\lim_{\ell \to \infty} \lambda_1^*(\ell) = \frac{4\pi^2 k^2}{\omega} .$$

Combining the results of 1), 2) and 3) we get the following: The maximum energy radiated into upper half-space (heat losses = 1), $\lambda_1^*(\ell)$ is a continuous non-decreasing function of ℓ and approaches $\frac{4\pi^2 k^2}{\omega}$ as $\ell \to \infty$. For $0 < \ell < \infty$, $\lambda_1(\ell) > \lambda_1^*(\ell) > \frac{\lambda_1(\ell)}{2}$ where $\lambda_1(\ell)$ is the maximum energy radiated into the full space.

10. Comment on a certain problem.

The ratio of energy radiated into upper half space to total radiated energy,

$$\frac{\mathcal{E}^*(f)}{\mathcal{E}(f)} = \frac{1}{2} + \frac{1}{2} \left\{ \frac{\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \frac{H_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}} \left(\alpha(u)\beta(v) - \alpha(v)\beta(u)\right) du\, dv}{\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \frac{J_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}} \left(\alpha(u)\alpha(v) + \beta(u)\beta(v)\right) du\, dv} \right\}$$

We set the problem of maximizing $\frac{\mathcal{E}^*(f)}{\mathcal{E}(f)}$ over the class of functions $f(s) = \alpha + i\beta(s) \in L^2(-\ell,\ell)$, $f(s) \equiv 0$ $|s| > \ell$. We therefore want to maximize

$$\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \frac{H_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}} \left(\alpha(u)\beta(v) - \alpha(v)\beta(u)\right) du\, dv$$

subject to the condition that

$$\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} \frac{J_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}} \left(\alpha(u)\alpha(v) + \beta(u)\beta(v)\right) du\, dv = \text{constant} .$$

Setting the first variations equal to zero gives the equations

$$(10.1)\quad \int_{-\ell}^{\ell} \left[\lambda \frac{J_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}} \alpha(u) - \frac{H_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}} \beta(u) \right] du = 0$$

$$(10.2)\quad \int_{-\ell}^{\ell} \left[\lambda \frac{J_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}} \beta(u) + \frac{H_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}} d(u) \right] du = 0 \quad .$$

We can rewrite (10.1) and (10.2) in the form

$$(10.3)\quad \int_{-\ell}^{\ell} \left[\lambda \frac{J_{3/2}\left[k(x-v)\right]}{\left[k(u-v)\right]^{3/2}} + i \frac{H_{3/2}\left[k(u-v)\right]}{\left[k(u-v)\right]^{3/2}} \right] \left[\alpha(u) + i\beta(u)\right] du = 0$$

If we apply $\int_{-\infty}^{\infty} e^{-ivs} \{\quad\} dv$

we get

$$(10.4)\quad (\lambda+1) \frac{1}{k^3} \sqrt{\frac{\pi}{2}} (k^2-s^2) \int_{-\ell}^{\ell} e^{ius} f(u)\, du = 0 \qquad 0 < s < k$$

and

$$(10.5)\quad (\lambda-1) \frac{1}{k^3} \sqrt{\frac{\pi}{2}} (k^2-s^2) \int_{-\ell}^{\ell} e^{ius} f(u)\, du = 0 \qquad -k < s < 0$$

Therefore $\int_{-\ell}^{\ell} e^{ius} f(u)\, du \equiv 0$ in a whole interval of s. Hence $f(u) \equiv 0$.

Therefore $\frac{\xi^*(f)}{\xi(f)}$ has no maximum.

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